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A NOTE ON THE TIME IDENTIFICATION NONLOCAL PROBLEM

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Abstract. This paper studies the time identification hyperbolic problems (IHPs) with periodic nonlocal conditions. Applying a new version of a operator approach, we establish the well-posedness of the IHP. A new absolute stable difference scheme for the approximate solution of this IHP is constructed. The well-posedness of this difference scheme is proved. Moreover, a numerical test is done.

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1 Introduction

Several IHPs have been well studied (Ashyralyev et al., 2003; Ashyralyev & Fattorini, 1992; Kal'menov & Sadybekov, 2017), and the literatures provided therein. The study of identification problems (IPs) for partial differential equation (PDE) play a significant role in engineering and applied sciences (Isakov, 2006; Kabanikhin & Krivorotko, 2015; Prilepko et al., 2000), and the resources provided therein). IPs for PDEs have been investigated (Ashyralyyev, 2014; Borukhov & Vabishchevich, 2000; Choulli & Yamamoto, 1999; Samarskii & Vabishchevich, 2008). The stability of the (IPs) in different formulations with several types of overdetermined conditions for hyperbolic and telegraph equations were considered (Ashyralyev & Çekiç, 2015; Ashyralyev & Emharab, 2019; Isgandarova, 2015; Kozhanov & Safiullova, 2010; Kozhanov & Safiullova , 2017; Kozhanov & Telesheva, 2017; Sabitov & Yunusova, 2012 and the literatures given therein). For the coefficients and solutions of PDEs several representations were presented (Anikonov, 1995; Anikonov, 1996; Anikonov & Neshchadim, 2011). They present such formulas for linear and nonlinear parabolic and hyperbolic differential problems (DPs). Finally, the authors (Grasselli et al., 1990; Grasselli et al., 1992; Grasselli, 1992) have been proposed significant examples for real applications of IHPs. Moreover, applications of IHPs can be study of physical phenomena like, e.g., viscoelasticity, electromagnetic wave propagation and heat conduction (Gurtin& Sternberg, 1962; Findly et al., 1976; Rabotnov, 1980; Renardy et al., 1987; Ben-Menahem & Singh, 2012; Gurtin & Pipkin, 1968).

In this work, we study the IHP with the desired functions u and p

$$\begin{cases} u_{tt}(t,x) - (a(x)u_x(t,x))_x + \delta u(t,x) \\ = p(t)\sigma(x) + f(t,x), 0 < x < l, 0 < t < T, \\ u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), 0 \le t \le T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), 0 \le x \le l, \\ \int_0^l u(t,x)\,dx = \omega(t), 0 \le t \le T. \end{cases} \tag{1}$$

Here $\sigma(x), \omega(t), \varphi(x)\ \psi(x)$ and $f(t,x)$ are given smooth functions. Suppose that

$$a(x) \ge a > 0, a(l) = a(0), \delta > 0,$$

$$\int_0^l \sigma(y)\,dy \ne 0, \sigma(0) = \sigma(l), \sigma'(0) = \sigma'(l). \tag{2}$$

The main aim of the present paper is to investigate the well-posedness of IHP (1) and construct absolute stable DSs for the numerical solution of the DP. The theorem on well-posedness of IHP (1) is established. The construction and investigation of absolute stable DSs for linear IHPs is important with different aspects: to study convergence of DSs and to consider the nonlinear DP(without Courant's conditions). It is our motivation for studying such type of problem. We present a new DS for approximate solution of inverse problem (1). As you see, it is an ill-posed problem. There are no such type of investigations in literature. For the numerical solution of IHP (1) a new absolute stable two-step DS is constructed. The theorem on stability estimates for the solution of this DS is proved. Moreover, the proof of the theorem is not based on classical tools of proof on stability of DSs. Numerical results are provided. Summarizing considered numerical experiment, we can conclude that all the theoretical statements of the previous sections can be verified.

2 Well-posedness of IHP(1)

Let $\mathbb{C}(\mathbb{H}) = \mathbb{C}([0,T], \mathbb{H})$ be a Banach space of all abstract continuous functions $v(y) \in \mathbb{H}$ determined on $[0,T]$ equipped with the norm

$$\|v\|_{\mathbb{C}(\mathbb{H})} = \max_{0 \leqslant y \leqslant T} \|v(y)\|_{\mathbb{H}}.$$

Let $\mathbb{L}_2[0,l]$ be the space of all square integrable functions $w(x)$ defined on $[0,l]$ and $\mathbb{W}_2^k[0,l], k = 1,2$ be Sobolev spaces equipped with norms

$$\|w\|_{\mathbb{L}_2[0,l]} = \left(\int_0^l w^2(z)\,dz\right)^{\frac{1}{2}},$$

$$\|w\|_{\mathbb{W}_2^1[0,l]} = \left(\int_0^l \left[w^2(z) + w_z^2(z)\right] dz\right)^{\frac{1}{2}},$$

$$\|w\|_{\mathbb{W}_2^2[0,l]} = \left(\int_0^l \left[w^2(z) + w_{zz}^2(z)\right] dz\right)^{\frac{1}{2}},$$

respectively. We consider the second order differential operator A determined by

$$Av = -(a(x)v_x(x))_x + \delta v(x) \tag{3}$$

in $\mathbb{L}_2[0,l]$ with domain $\mathbb{D}(A) = \{v : v, v'' \in \mathbb{L}_2[0,l], v(0) = v(l), v'(0) = v'(l)\}$ dense in $\mathbb{L}_2[0,l]$. It is clear that A is the positive-definite self-adjoint operator(PDSAO) in $\mathbb{L}_2[0,l]$.

In the present paper, we will introduce the notation $M(\delta, \sigma, ...)$ to stress the fact that the constant depends only on $\delta, \sigma, ...$, which may differ in time and thus is not a subject of precision.

Theorem 1. *Let* $\varphi \in \mathbb{W}_2^2[0,l]$ *and* $\psi \in \mathbb{W}_2^1[0,l]$. *Suppose that* $f,\ f_t \in \mathbb{C}(\mathbb{L}_2[0,l])$ *and* $\omega, \omega'' \in \mathbb{C}[0,T]$. *Then the IHP has a unique solution* $(u,p) \in \mathbb{C}(\mathbb{L}_2[0,l]) \times \mathbb{C}[0,T]$.

Proof. Assume that $w(t,x)$ be the solution of the initial-nonlocal boundary value DP

$$\begin{cases} w_{tt}(t,x) - (a(x) w_x(t,x))_x + \delta w(t,x) \\ = f(t,x) + \mu(t)\left[(a(x)\sigma_x(x))_x - \delta\sigma(x)\right], \\ 0 < x < l, 0 < t < T, \\ w(t,0) = w(t,l), w_x(t,0) = w_x(t,l), 0 \le t \le T, \\ w(0,x) = \varphi(x), w_t(0,x) = \psi(x), 0 \le x \le l \end{cases} \tag{4}$$

and $\mu(t)$ be the function determining by

$$\mu(t) = \int_0^t (t-z)\, p(z)\, dz, \mu(0) = \mu'(0) = 0. \tag{5}$$

Then,

$$u(t,x) = w(t,x) + \mu(t)\sigma(x). \tag{6}$$

Using the integral condition in (1) and formula (6), we can obtain

$$\mu(t) = \frac{1}{Q}\left(\omega(t) - \int_0^l w(t,p)\, dp\right), Q = \int_0^l \sigma(p)\, dp. \tag{7}$$

Since $p(t) = \mu''(t)$, we obtain

$$p(t) = \frac{1}{Q}\left(\omega''(t) - \int_0^l w_{tt}(t,p)\, dp\right). \tag{8}$$

Therefore, the following theorem will be complete the proof of Theorem 1. □

Theorem 2. *Under assumptions of Theorem 1, the initial-boundary value DP (4) has a unique solution* $w \in \mathbb{C}(\mathbb{L}_2[0,l])$.

Proof. The DP (4) is equivalent to the integral equation

$$w(t,x) = c(t)\varphi(x) + s(t)\psi(x) \tag{9}$$

$$+ \int_0^t s(t-z)\left\{ f(z,x) - \frac{1}{Q}\left(\omega(z) - \int_0^l w(z,p)\, dp\right) Aq(x)\right\} dz$$

in $\mathbb{C}[0,T] \times \mathbb{C}[0,l]$. Here, $c(t)$ and $s(t)$ are operator-functions generated by the operator A and defined by formulas

$$c(t)u = \frac{e^{iA^{\frac{1}{2}}t} + e^{-iA^{\frac{1}{2}}t}}{2}u, \quad s(t)u = \int_0^t c(y)u dy. \tag{10}$$

Let us give estimates (see, Ashyralyev & Fattorini, 1992) that will be needed below

$$\|A^{-\frac{1}{2}}\|_{\mathbb{L}_2[0,l] \to \mathbb{L}_2[0,l]} \le \delta^{-\frac{1}{2}}, \ \|s(t)\|_{\mathbb{L}_2[0,l] \to \mathbb{L}_2[0,l]} \le t,$$

$$\|c(t)\|_{\mathbb{L}_2[0,l] \to \mathbb{L}_2[0,l]} \le 1, \quad \|A^{\frac{1}{2}}s(t)\|_{\mathbb{L}_2[0,l] \to \mathbb{L}_2[0,l]} \le 1. \tag{11}$$

Then, the recursive formula for the solution of DP (4) is defined by

$$w_j(t,x) = c(t)\varphi(x) + s(t)\psi(x) \tag{12}$$

$$+\int_0^t s(t-z)\left\{f(z,x)-\frac{1}{Q}\left(\omega(z)-\int_0^l w_{j-1}(z,p)\,dp\right)A\sigma(x)\right\}dz,$$

$$w_0(t,x)=c(t)\,\varphi(x)+s(t)\,\psi(x)\,,j\geq 1.$$

Therefore,

$$w(t,x)=w_0(t,x)+\sum_{i=0}^{\infty}\left(w_{i+1}(t,x)-w_i(t,x)\right). \tag{13}$$

Applying estimates (11), we can obtain

$$\|w_0(t,.)\|_{\mathbb{L}_2[0,l]}\leqslant\|\varphi\|_{\mathbb{L}_2[0,l]}+T\,\|\psi\|_{\mathbb{L}_2[0,l]}=M_0$$

for every $t\in[0,T]$. Applying formula (12), we get

$$w_1(t,x)-w_0(t,x)$$

$$=\int_0^t s(t-y)\left\{f(z,x)-\frac{1}{Q}\left(\omega(z)-\int_0^l w(z,p)\,dp\right)Aq(x)\right\}dz,$$

$$w_{j+1}(t,x)-w_j(t,x)$$

$$=\int_0^t s(t-z)\left\{\frac{\int_0^l\left(w_j(z,p)-w_{j-1}(z,p)\right)dp}{Q}Aq(x)\right\}dz,j\geq 2.$$

Applying estimates (11), we can get

$$\|w_1(t,\cdot)-w_0(t,\cdot)\|_{\mathbb{L}_2[0,l]}\leq\int_0^t\left\|\left\{A^{-\frac{1}{2}}f(z,\cdot)-\frac{1}{Q}\left(\omega(z)-\int_0^l w(z,p)\,dp\right)A^{\frac{1}{2}}q(\cdot)\right\}\right\|_{\mathbb{L}_2[0,l]}dz$$

$$\leq\left\{\frac{1}{\sqrt{\delta}}\max_{0\leq z\leq T}\|f(z,\cdot)\|_{\mathbb{L}_2[0,l]}\right.$$

$$\left.+\frac{1}{|Q|}\left[\max_{0\leq z\leq T}|\omega(z)|+\sqrt{l}\max_{0\leq z\leq T}\|w_0(z,.)\|_{\mathbb{L}_2[0,l]}\right]\left\|A^{\frac{1}{2}}\sigma(\cdot)\right\|_{\mathbb{L}_2[0,l]}\right\}t\leq Mt,$$

$$\|w_{j+1}(t,\cdot)-w_j(t,\cdot)\|_{\mathbb{L}_2[0,l]}\leq\int_0^t\left\|\left\{\frac{\int_0^l w_j(z,p)-w_{j-1}(z,p)\,dp}{Q}A^{\frac{1}{2}}\sigma(x)\right\}\right\|_{\mathbb{L}_2[0,l]}dz$$

$$\leq\frac{\sqrt{l}}{|Q|}\left\|A^{\frac{1}{2}}\sigma(\cdot)\right\|_{\mathbb{L}_2[0,l]}\int_0^t\|w_j(z,\cdot)-w_{j-1}(z,\cdot)\|_{\mathbb{L}_2[0,l]}\,dz$$

$$\leq K_1\int_0^t\|w_j(z,\cdot)-w_{j-1}(z,\cdot)\|_{\mathbb{L}_2[0,l]}\,dz,j\geq 2$$

for every $t\in[0,T]$. Applying the triangle inequality, we can obtain

$$\|w_1(t,\cdot)\|_{\mathbb{L}_2[0,l]}\leq M_0+Mt$$

for any $t\in[0,T]$. Moreover,

$$\|w_2(t,\cdot)-w_1(t,\cdot)\|_{\mathbb{L}_2[0,l]}\leq K_1\int_0^t\|w_1(z,\cdot)-w_0(z,\cdot)\|_{\mathbb{L}_2[0,l]}\,dz$$

$$\leq K_1M\int_0^t zdz=K_1M\frac{t^2}{2}$$

for every $t \in [0,T]$. Let

$$\|w_j(t,\cdot) - w_{j-1}(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le \frac{M}{K_1}\frac{(K_1 t)^j}{j!}$$

for any $t \in [0,T]$. Then, we have that

$$\|w_{j+1}(t,\cdot) - w_j(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le K_1 \int_0^t \|w_j(z,\cdot) - w_{j-1}(z,\cdot)\|_{\mathbb{L}_2[0,l]} dz$$

$$\le K_1 \int_0^t \frac{M}{K_1}\frac{(K_1 z)^j}{j!} dy = \frac{M}{K_1}\frac{(K_1 t)^{j+1}}{(j+1)!}$$

for any $t \in [0,T]$. Therefore, using the mathematical induction

$$\|w_{j+1}(t,\cdot) - w_j(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le \frac{M}{K_1}\frac{(K_1 t)^{j+1}}{(j+1)!}$$

and

$$\|w_{j+1}(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le M_0 + \frac{M}{K_1}\frac{Kt}{1!} + ... + \frac{M}{K}\frac{(Kt)^{j+1}}{(j+1)!}$$

for all $t \in [0,T]$ and j, $j \geqslant 2$. Applying formula (13) and these estimates, we get

$$\|w(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le \|w_0(t,\cdot)\|_{\mathbb{L}_2[0,l]} + \sum_{i=0}^{\infty} \|w_{i+1}(t,\cdot) - w_i(t,\cdot)\|_{\mathbb{L}_2[0,l]}$$

$$\le M_0 + \sum_{i=0}^{\infty} \frac{M}{K}\frac{(K_1 t)^{i+1}}{(i+1)!} \le M_0 + \frac{M}{K_1} e^{K_1 t}$$

for any $t \in [0,T]$ which proves the existence of a bounded solution of DP (4) in $\mathbb{C}(\mathbb{L}_2[0,l])$. Now, we will prove uniqueness of solution of problem (4). Suppose that there is a second bounded solution $v(t,x)$ of (DP) (4) and $v(t,x) \neq w(t,x)$. We put $r(t,x) = v(t,x) - w(t,x)$. Therefore, we have that

$$r(t,x) = \int_0^t s(t-y)\frac{\int_0^l r(y,p)dp}{Q} A\sigma(x)\, dy$$

for $z(t,x)$. Applying estimates (11), we get

$$\|r(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le \frac{\sqrt{l}}{|Q|}\left\|A^{\frac{1}{2}} q(\cdot)\right\|_{\mathbb{L}_2[0,l]} \int_0^t \|r(y,\cdot)\|_{\mathbb{L}_2[0,l]} dy$$

$$\le K_1 \int_0^t \|r(y,\cdot)\|_{\mathbb{L}_2[0,l]} dy$$

for any $t \in [0,T]$. Therefore, using the integral inequality, we get

$$\|r(t,\cdot)\|_{\mathbb{L}_2[0,l]} \le 0$$

for any $t \in [0,T]$ and $r(t,x) = 0$ which proves the uniqueness of a bounded solution of DP (4) in $\mathbb{C}(\mathbb{L}_2[0,l])$.

Finally, taking the first and second order derivatives of (9) with respect to t, we can obtain

$$w_t(t,x) = -s(t) A\varphi(x) + c(t)\psi(x)$$

$$+ \int_0^t c(t-y)\left\{ f(y,x) - \frac{1}{Q}\left(\omega(y) - \int_0^l w(y,p)\,dp\right) A\sigma(x)\right\} dy,$$

$$w_{tt}(t,x) = -c(t)\,A\varphi(x) - A^{\frac{1}{2}}s(t)\,A^{\frac{1}{2}}\psi(x)$$
$$+c(t)\left\{f(0,x) - \frac{1}{Q}\left(\omega(0) - \int_0^l \varphi(p)\,dp\right)A\sigma(x)\right\}$$
$$+\int_0^t c(t-y)\left\{f_y(y,x) - \frac{1}{Q}\left(\omega'(y) - \int_0^l w_y(y,p)\,dp\right)A\sigma(x)\right\}dy.$$

Therefore, under assumptions of Theorem 1 and $w \in \mathbb{C}(\mathbb{L}_2[0,l])$ they follow $w_t,\ w_{tt}, Aw \in \mathbb{C}(\mathbb{L}_2[0,l])$. Theorem 2 is established. $\square$

Now, let us state the stability result.

Theorem 3. *Assume that the assumptions of Theorem 1 hold. The solution of IHP (1) obeys the stability estimate:*

$$\|u_{tt}\|_{\mathbb{C}(\mathbb{L}_2[0,l])} + \|u\|_{\mathbb{C}\left(\mathbb{W}_2^2[0,l]\right)} + \|p\|_{\mathbb{C}[0,T]} \tag{14}$$
$$\leqslant M(\delta,\sigma)\left[\|\varphi\|_{\mathbb{W}_2^2[0,l]} + \|\psi\|_{\mathbb{W}_2^1[0,l]} + \|f(0,.)\|_{\mathbb{L}_2[0,l]}\right.$$
$$\left.+\|f_t\|_{\mathbb{C}(\mathbb{L}_2[0,l])} + \left\|\omega''\right\|_{\mathbb{C}[0,T]}\right].$$

Proof. Applying formula (8) and $Q \neq 0$, we get the estimate

$$|p(t)| \leq M_1(\delta,\sigma)\left[\left|\omega''(t)\right| + \|w_{tt}\|_{\mathbb{C}(\mathbb{L}_2[0,l])}\right] \tag{15}$$

for every $t, t \in [0,T]$ and

$$\|p\|_{\mathbb{C}[0,T]} \leqslant M_1(\delta,\sigma)\left[\left\|\omega''\right\|_{\mathbb{C}[0,T]} + \|w_{tt}\|_{\mathbb{C}(\mathbb{L}_2[0,l])}\right]. \tag{16}$$

Now, applying formula (6), we can write

$$u_{tt}(t,x) = w_{tt}(t,x) + p(t)\,\sigma(x).$$

By the triangle inequality, this formula yields us

$$\|u_{tt}\|_{\mathbb{C}(\mathbb{L}_2[0,l])} \leqslant \|w_{tt}\|_{\mathbb{C}(\mathbb{L}_2[0,l])} + \|p\|_{\mathbb{C}[0,T]}\,\|\sigma\|_{\mathbb{L}_2[0,l]}. \tag{17}$$

Then, the proof of estimate (14) is based on equation (1), estimates (16), (17) and on the following stability estimate

$$\|w_{tt}\|_{\mathbb{C}(\mathbb{L}_2[0,l])} \leqslant M_2(\sigma,a)\left[\|\varphi\|_{\mathbb{W}_2^2[0,l]} + \|\psi\|_{\mathbb{W}_2^1[0,l]}\right. \tag{18}$$
$$\left.+\|f(0,.)\|_{\mathbb{L}_2[0,l]} + \|f_t\|_{\mathbb{C}(\mathbb{L}_2[0,l])} + \left\|\omega''\right\|_{\mathbb{C}[0,T]}\right].$$

It is clear that the mixed DP (4) can be written as the initial value problem

$$w_{tt}(t) + Aw(t) + \mu(t)\,A\sigma = f(t),\ t \in (0,T);\ w(0) = \varphi, w'(0) = \psi \tag{19}$$

in a Hilbert space $\mathbb{H} = \mathbb{L}_2[0,l]$ with the PDSAO A determining by (3). Therefore, applying the results of Lemma 2.2 in (Ashyralyev & Emharab, 2019), we can obtain the following stability estimate

$$\|w_{tt}(t)\|_{\mathbb{H}} \leqslant \|A\varphi\|_{\mathbb{H}} + \left\|A^{\frac{1}{2}}\psi\right\|_{\mathbb{H}} + \|f(0)\|_{\mathbb{H}} \tag{20}$$
$$+T\,\|f_t\|_{\mathbb{C}(\mathbb{H})} + M_3(\sigma,\delta)\int_0^t \left|\mu''(z)\right|dz$$

for any $t \in [0,T]$. Then, applying estimates (15), (20), and the integral inequality, we conclude that the following stability estimate

$$\|\omega_{tt}\|_{\mathbb{L}_2[0,l]} \leq M_4(\sigma,\delta)\left\{\|\varphi\|_{\mathbb{W}_2^2[0,l]} + \|\psi\|_{\mathbb{W}_2^1[0,l]} + \|f(0,\cdot)\|_{\mathbb{L}_2[0,l]}\right.$$

$$\left. +T\,\|f_t\|_{\mathbb{C}(\mathbb{L}_2[0,l])} + \left\|\omega''\right\|_{\mathbb{C}[0,T]}\right\} e^{M_2(\delta,\sigma)M_3(\delta,\sigma)}.$$

is satisfied for the solution of DP (4) for every $t, t \in [0,T]$. Estimate (18) is proved. Theorem 3 is established. □

3 The absolute stable two-step DS. Well-posedness

To state our results, we consider a normed space $\mathbb{C}_\tau(\mathbb{H}) = \mathbb{C}\left([0,T]_\tau, \mathbb{H}\right)$ of all abstract mesh functions $f^\tau = \{f(t_m)\}_{k=0}^{K}$ with values in $\mathbb{H}$ defined on the uniform mesh space

$$[0,T]_\tau = \{t_m = m\tau, m = 0,1,...,K, K\tau = T\}, \tau > 0$$

equipped with the norm

$$\|f^\tau\|_{\mathbb{C}_\tau(\mathbb{H})} = \max_{0\leqslant m\leqslant K} \|f(t_m)\|_{\mathbb{H}}.$$

Let $\mathbb{L}_{2h} = \mathbb{L}_2[0,l]_h$ and $\mathbb{W}_{2h}^k = \mathbb{W}_2^p[0,l]_h, p = 1,2$ be normed spaces of all mesh functions $v^h(x) = \{v_i\}_{i=0}^{M}$ defined on the uniform mesh space

$$[0,l]_h = \{x_i = ih, i = 0,1,...,M, Mh = l\},$$

equipped with norms

$$\left\|v^h\right\|_{\mathbb{L}_{2h}} = \left\{\sum_{i=0}^{M} v_i^2 h\right\}^{\frac{1}{2}},$$

$$\left\|v^h\right\|_{\mathbb{W}_{2h}^1} = \left\{\sum_{i=0}^{M} v_i^2 h + \sum_{i=1}^{M}\left|\frac{1}{h}(v_i - v_{i-1})\right|^2 h\right\}^{\frac{1}{2}},$$

$$\left\|v^h\right\|_{\mathbb{W}_{2h}^2} = \left\{\sum_{i=0}^{M} v_i^2 h + \sum_{i=1}^{M-1}\left|\frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1})\right|^2 h\right\}^{\frac{1}{2}},$$

respectively. We denote the second order difference operator A_h determined by formula

$$A_h\varphi^h(x) = \left\{-\frac{1}{h^2}\left(a_{m+1}(\varphi_{m+1} - \varphi_m) - a_m(\varphi_m - \varphi_{m-1})\right) + \delta\varphi_m\right\}_{m=1}^{M-1} \tag{21}$$

acting in the space of grid functions $\varphi^h(x)$ satisfying the nonlocal conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. Here $a_m = a(x_m)$ for $0 \leq m \leq M$. It is clear that A_h is the PDSAO in $\mathbb{L}_{2h}$.

We present the first order of accuracy absolute stable two-step DS

$$\begin{cases} \frac{1}{\tau^2}\left(u_n^{k+1}-2u_n^k+u_n^{k-1}\right)-\frac{1}{h^2}\left(a_{n+1}\left(u_{n+1}^{k+1}-u_n^{k+1}\right)-a_n\left(u_n^{k+1}-u_{n-1}^{k+1}\right)\right) \\ +\delta u_n^{k+1}=p_k\sigma_n+f\left(t_k,x_n\right), t_k=k\tau, x_n=nh, \\ 1\leqslant k\leqslant K-1, 1\leqslant n\leqslant M-1, \\ u_0^{k+1}=u_M^{k+1}, u_1^{k+1}-u_0^{k+1}=u_M^{k+1}-u_{M-1}^{k+1}, 1\leqslant k\leqslant K-1, \\ u_n^0=\varphi\left(x_n\right), \dfrac{1}{\tau}\left(u_n^1-u_n^0\right)=\psi\left(x_n\right), 0\leqslant n\leqslant M, \\ \sum_{i=1}^{M-1}u_i^{k+1}h=\omega\left(t_{k+1}\right), -1\leqslant k\leqslant K-1 \end{cases} \tag{22}$$

for the numerical solution of the IHP (1). Suppose that

$$Q_h=\sum_{i=1}^{M-1}\sigma_i h\neq 0, \sigma_0=\sigma_M, \sigma_1-\sigma_0=\sigma_M-\sigma_{M-1}.$$

Now, let us investigate the stability of DS (22).

Theorem 4. *The solution of DS (22) satisfies the stability estimate:*

$$\left\|\left\{\frac{1}{\tau^2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})}+\left\|\left\{u_{k+1}^h\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau\left(\mathbb{W}_{2h}^2\right)}+\left\|\{p_k\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau} \tag{23}$$

$$\leqslant M_5(\sigma,\delta)\left[\left\|\varphi^h\right\|_{\mathbb{W}_{2h}^2}+\left\|\psi^h\right\|_{\mathbb{W}_{2h}^1}+\left\|f_1^h\right\|_{\mathbb{L}_{2h}}\right.$$

$$\left.+\left\|\left\{\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\}_{k=2}^{K-1}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})}+\left\|\left\{\frac{1}{\tau^2}\left(\omega_{k+1}-2\omega_k+\omega_{k-1}\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau}\right],$$

where $f_k^h(x)=\{f(t_k,x_n)\}_{n=0}^M, 1\leq k\leq K-1.$

Proof. Denoted as

$$u_n^k=w_n^k+\mu_k\sigma_n, \tag{24}$$

where $\{\mu_k\}_{k=0}^K$ is the grid function determined by

$$\mu_{k+1}=\sum_{i=0}^{k-1}(k-i)\,p_{i+1}\tau^2, 1\leqslant k\leqslant K-1, \mu_0=\mu_1=0 \tag{25}$$

and $\left\{\{w_n^k\}_{k=0}^K\right\}_{n=0}^M$ is the grid function determined as the solution of DS

$$\begin{cases} \frac{1}{\tau^2}\left(w_n^{k+1}-2w_n^k+w_n^{k-1}\right)-\frac{1}{h}\left(a_{n+1}\frac{w_{n+1}^{k+1}-w_n^{k+1}}{h}-a_n\frac{w_n^{k+1}-w_{n-1}^{k+1}}{h}\right)+\delta w_n^{k+1} \\ =f\left(t_k,x_n\right)+\mu_{k+1}\frac{1}{h^2}\left[a_{n+1}\left(\sigma_{n+1}-\sigma\right)-a_n\left(\sigma_n-\sigma_{n-1}\right)-\delta h^2\sigma_n\right], \\ 1\leqslant k\leqslant K-1, 1\leqslant n\leqslant M-1, \\ w_0^{k+1}=w_M^{k+1}, w_1^{k+1}-w_0^{k+1}=w_M^{k+1}-w_{M-1}^{k+1}, -1\leqslant k\leqslant K-1, \\ w_n^0=\varphi\left(x_n\right), w_n^1=\varphi\left(x_n\right)+\tau\psi\left(x_n\right), 0\leqslant n\leqslant M. \end{cases} \tag{26}$$

Applying the overdetermined condition $\sum_{i=1}^{M-1} u_i^{k+1}h = \omega\left(t_{k+1}\right)$ and substitution (24), one can obtain that

$$\mu_{k+1} = \frac{1}{Q_h}\left(\omega_{k+1} - \sum_{i=1}^{M-1} w_i^{k+1}h\right). \tag{27}$$

Using formulas $p_k = \frac{\mu_{k+1}-2\mu_k+\mu_{k-1}}{\tau^2}$ and (27), we obtain

$$p_m = \frac{1}{\tau^2 Q_h}\left(\omega_{m+1} - 2\omega_m + \omega_{m-1} - \sum_{i=1}^{M-1}\left(w_i^{m+1} - 2w_i^m + w_i^{m-1}\right)h\right).$$

Then, applying the Cauchy–Schwartz inequality, we can write

$$|p_m| \leqslant M_6\left(\sigma\right)\left[\left|\frac{1}{\tau^2}\left(\omega_{m+1} - 2\omega_m + \omega_{m-1}\right)\right| + \left\|\frac{1}{\tau^2}\left(w_{m+1}^h - 2w_m^h + w_{m-1}^h\right)\right\|_{\mathbb{L}_{2h}}\right] \tag{28}$$

for every $1 \leqslant m \leqslant K-1$. Therefore,

$$\left\|\left\{p_k\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau} \leq M_6\left(\sigma\right)\left[\left\|\left\{\frac{1}{\tau^2}\left(\omega_{k+1} - 2\omega_k + \omega_{k-1}\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau}\right. \tag{29}$$

$$\left. + \left\|\left\{\frac{1}{\tau^2}\left(w_{k+1}^h - 2w_k^h + w_{k-1}^h\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})}\right].$$

Now, using substitution (24), we get

$$\left\|\left\{\left\{\frac{1}{\tau^2}\left(u_i^{k+1} - 2u_i^k + u_i^{k-1}\right)\right\}_{k=1}^{K-1}\right\}_{i=0}^{M}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})} \tag{30}$$

$$\leqslant \left\|\left\{\left\{\frac{1}{\tau^2}\left(w_i^{k+1} - 2w_i^k + w_i^{k-1}\right)\right\}_{k=1}^{K-1}\right\}_{i=0}^{M}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})}$$

$$+ \left\|\left\{p_k\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau}\left\|\left\{\sigma_i\right\}_{i=0}^{M}\right\|_{\mathbb{L}_{2h}}.$$

Therefore, the proof of estimate (23) is based on equation (22), on estimates (29), (30) and

$$\left\|\left\{\frac{1}{\tau^2}\left(w_{k+1}^h - 2w_k^h + w_{k-1}^h\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})} \tag{31}$$

$$\leqslant M_7(q)\left[\left\|\varphi^h\right\|_{\mathbb{W}_{2h}^2} + \left\|\psi^h\right\|_{\mathbb{W}_{2h}^1} + \left\|f_1^h\right\|_{\mathbb{L}_{2h}}\right.$$

$$\left. + \left\|\left\{\frac{1}{\tau}\left(f_k^h - f_{k-1}^h\right)\right\}_{k=2}^{K-1}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})} + \left\|\left\{\frac{1}{\tau^2}\left(\omega_{k+1} - 2\omega_k + \omega_{k-1}\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau}\right]$$

for the solution of DS (26).

It is clear that DS (26) can be written as the abstract DS

$$\begin{cases} \frac{1}{\tau^2}\left(w_{k+1} - 2w_k + w_{k-1}\right) + Aw_{k+1} + \mu_{k+1}A\sigma = f_k, \\ 1 \leqslant k \leqslant K-1, w_0 = \varphi, w_1 = \varphi + \tau\psi \end{cases} \tag{32}$$

in a Hilbert space $\mathbb{H}=\mathbb{L}_{2h}$ with the PDSAO A_h defined by (21).
Therefore, using the results of Lemma 3.2 in (Ashyralyev & Emharab, 2019), we get the following stability estimate

$$\left\|\frac{1}{\tau^2}w_{k+1}-2w_k+w_{k-1}\right\|_{\mathbb{H}}\leqslant\|A\varphi\|_{\mathbb{H}}+\left\|A^{\frac{1}{2}}\psi\right\|_{\mathbb{H}}+\|f_1\|_{\mathbb{H}} \tag{33}$$

$$+T\left\|\left\{\frac{1}{\tau}\left(f_k-f_{k-1}\right)\right\}_{k=2}^{K-1}\right\|_{\mathbb{C}(\mathbb{H})}+\sum_{s=1}^{k}\left|\frac{1}{\tau^2}\left(\mu_{s+1}-2\mu_s+\mu_{s-1}\right)\right|\tau\left\|A^{\frac{1}{2}}\sigma\right\|_{\mathbb{H}}$$

for every $1\leqslant k\leqslant K-1$.

Then, applying estimates (28), (33), and the difference analogue of Gronwall's inequality, we conclude that for the solution DS (26) the following stability estimate

$$\left\|\frac{1}{\tau^2}w_{k+1}^h-2w_k^h+w_{k-1}^h\right\|_{\mathbb{L}_{2h}}\leqslant\frac{M_{10}(\sigma,\delta)}{1-M_9(\sigma)M_6(\sigma)\tau}\left\{M_8(\sigma)\left[\left\|\varphi^h\right\|_{\mathbb{W}_{2h}^2}+\left\|\psi^h\right\|_{\mathbb{W}_{2h}^1}\right]\right.$$

$$+\left\|f_1^h\right\|_{\mathbb{L}_{2h}}+T\left\|\left\{\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\}_{k=2}^{K-1}\right\|_{\mathbb{C}_\tau(\mathbb{L}_{2h})}+M_9(\sigma)M_6(\sigma)T$$

$$\left.\times\left\|\left\{\frac{1}{\tau^2}\left(\omega_{k+1}-2\omega_k+\omega_{k-1}\right)\right\}_{k=1}^{K-1}\right\|_{\mathbb{C}_\tau}\right\}e^{(k-1)\tau\frac{M_9(\sigma)M_6(\sigma)}{1-M_9(\sigma)M_6(\sigma)\tau}}.$$

holds for every $1\leq k\leq K-1$.Estimate (31) is proved. Theorem 2 is established. □

4 Numerical Results

We will consider the IHP with the exact solution $(u,p)=\left(e^{-2t}\left(1+\sin 2x\right),e^{-2t}\right)$

$$\begin{cases} u_{tt}-u_{xx}-4p\left(t\right)\left(1+\sin 2x\right)=4e^{-2t}\sin 2x,\\ 0<x<\pi,\ 0<t<1,\\ u\left(t,0\right)=u\left(t,\pi\right),u_x\left(t,0\right)=u_x\left(t,\pi\right),\ 0\leq t\leq 1,\\ u\left(0,x\right)=1+\sin 2x,\ u_t\left(0,x\right)=-2\left(1+\sin 2x\right),\ 0\leq x\leq\pi,\\ \int_0^\pi u\left(t,x\right)dx=\pi e^{-2t},\ 0\leq t\leq 1. \end{cases} \tag{34}$$

Applying DS (22) for DP (34) and formula (21), we get the following DS

$$\begin{cases} \frac{1}{\tau^2}\left(u_m^{j+1}-2u_m^j+u_m^{j-1}\right)-\frac{1}{h^2}\left(u_{m+1}^{j+1}-2u_m^{j+1}+u_{m-1}^{j+1}\right)-4p_j\left(1+\sin 2x_m\right)\\ =4e^{-2t_{j+1}}\sin 2x_m,\ t_j=j\tau,\ x_m=mh,\\ 1\leqslant j\leqslant K-1,1\leqslant m\leqslant M-1,\\ u_0^{j+1}=u_M^{j+1},u_1^{j+1}-u_0^{j+1}=u_M^{j+1}-u_{M-1}^{j+1},\ 1\leqslant j\leqslant K-1,\\ u_m^0=1+\sin 2x_m,\frac{1}{\tau}\left(u_m^1-u_m^0\right)=-2\left(1+\sin 2x_m\right),\ 0\leqslant m\leqslant M,\\ \pi e^{-2t_{j+1}}=\sum_{i=1}^{M-1}u_i^{j+1}h,\ 1\leqslant j\leqslant K-1. \end{cases} \tag{35}$$

For obtaining the solution of IP (35), we will apply the following substitution

$$u_m^j=w_m^j+4\mu_j\left(1+\sin 2x_m\right),0\leq j\leq K,0\leq m\leq M, \tag{36}$$

where

$$\mu_{j+1}=\frac{1}{4p}\left(\pi e^{-2t_{j+1}}-\sum_{i=1}^{M-1}w_i^{j+1}h\right),\rho=\sum_{i=1}^{M-1}\left(1+\sin 2x_i\right)h \tag{37}$$

for every $-1 \leq j \leq K-1$ and $\left\{\left\{w_m^j\right\}_{j=0}^K\right\}_{m=0}^M$ be the mesh function determined as the solution of DS

$$\begin{cases} \frac{w_m^{j+1}-2w_m^j+w_m^{j-1}}{\tau^2} - \frac{w_{m+1}^{j+1}-2w_m^{j+1}+w_{m-1}^{j+1}}{h^2} \\ +\frac{1}{\rho}\left(\sum_{i=1}^{M-1} w_i^{j+1}h\right)\sin 2x_m \frac{2}{h^2}(\cos 2h-1) \\ = \left[\frac{1}{h^2\rho}\pi(\cos 2h-1)+2\right]2e^{-2t_{j+1}}\sin 2x_m, \\ 1 \leqslant j \leqslant K-1, 1 \leqslant m \leqslant M-1, \\ w_0^{j+1} = w_M^{j+1}, w_1^{j+1}-w_0^{j+1} = w_M^{j+1}-w_{M-1}^{j+1}, -1 \leqslant j \leqslant K-1, \\ w_m^0 = 1+\sin 2x_m, \frac{1}{\tau}\left(w_m^1 - w_m^0\right) = -2\left(1+\sin 2x_m\right), 0 \leqslant m \leqslant M. \end{cases} \tag{38}$$

Here and in future $\rho = \sum_{i=1}^{M-1}(1+\sin 2x_i)h$, Therefore, solution of IHP (35) contains three stages. At the first stage, we find the solution of DS (38). For obtaining it, we write DS (38) in matrix form as the second order difference problem

$$\begin{cases} Aw^{j+1}+Bw^j+Cw^{j-1} = f^j, 1 \leqslant j \leqslant K-1, \\ w^0 = \{1+\sin 2x_m\}_{m=0}^M, w^1 = (1-2\tau)w^0, \end{cases} \tag{39}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & -1 \\ b & a+c_1 & b+c_1 & \cdot & c_1 & 0 \\ 0 & b+c_2 & a+c_2 & \cdot & c_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & c_{M-1} & c_{M-1} & \cdot & a+c_{M-1} & b \\ -1 & 1 & 0 & \cdot & 1 & -1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 \\ 0 & e & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & e & 0 \\ 0 & 0 & \cdot & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)} \quad C = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 \\ 0 & g & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & g & 0 \\ 0 & 0 & \cdot & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$f^j = \begin{bmatrix} 0 \\ f_1^j \\ \cdot \\ f_{M-1}^j \\ 0 \end{bmatrix}_{(M+1)\times 1} \quad w^s = \begin{bmatrix} w_0^s \\ w_1^s \\ \cdot \\ w_{M-1}^s \\ w_M^s \end{bmatrix}_{(M+1)\times 1}, \text{ for } s = j, j\pm 1.$$

Here,

$$a = \frac{1}{\tau^2}+\frac{2}{h^2}, b = -\frac{1}{h^2}, c_m = \frac{2(\cos 2h-1)}{hd}\sin 2x_m,$$
$$d = \sum_{i=1}^{M-1}(1+\sin 2x_i)h, e = -\frac{2}{\tau^2}, g = \frac{1}{\tau^2},$$

$$f_m^j = \left[\frac{1}{h^2\rho}2\pi(\cos 2h-1)+2\right]2e^{-2t_{j+1}}\sin 2x_m,$$

$$1 \leqslant j \leqslant K-1, 1 \leqslant m \leqslant M-1.$$

Then, we have the second order difference problem (39) in j with matrix coefficients. Since $w^0$and $w^1$are provided, we get

$\left\{\left\{w_m^j\right\}_{j=0}^K\right\}_{m=0}^M$ by (39). Then, using (25), we get

$$p_j = \frac{1}{\tau^2}\left(\mu_{j+1} - 2\mu_j + \mu_{j-1}\right), 1 \leq j \leq K-1. \tag{40}$$

We will obtain $\{p_j\}_{j=1}^{K-1}$ by formulas (37) and (40) on the second stage. We can obtain $\left\{\left\{u_m^j\right\}_{j=0}^K\right\}_{m=0}^M$ by formulas (36) and (37) in the third stage. The errors between the exact solution (u, p) of (34) at (t_j, x_m) and approximate solution $\left(u_m^j, p_j\right)$ of DS (35) are computed by

$$E_u = \max_{0 \leqslant j \leqslant K}\left(\sum_{m=0}^M \left|u\left(t_j, x_m\right) - u_m^j\right|^2 h\right)^{\frac{1}{2}}, \tag{41}$$

$$E_p = \max_{1 \leqslant j \leqslant K-1}\left|p\left(t_j\right) - p_j\right|.$$

Now, let us give the obtained numerical results.

Table 1: Numerical results

Errors/N = M	**20**	**40**	**80**	**160**
E_u	0.0560	0.0289	0.0147	0.0074
E_p	0.0476	0.0244	0.0123	0.0062

The obtained results show that if N and M are doubled, the value of errors of solution of the DS (35) decreases by a factor of approximately 1/2 (see Table 1 up). Moreover, in Figures 1, 2 and 3 below, plots for errors of u and p that decrease by a factor of approximately $\frac{1}{2}$ for different values of the steps, respectively.

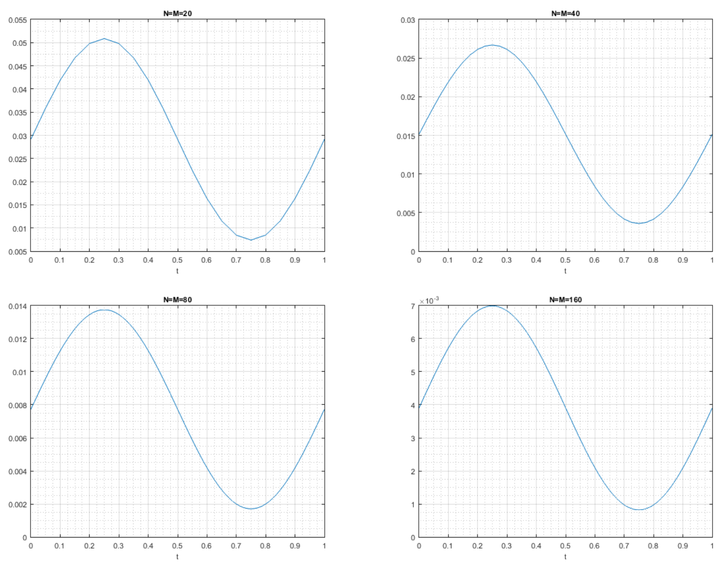

Figure 1: Graphs for errors of u in t.

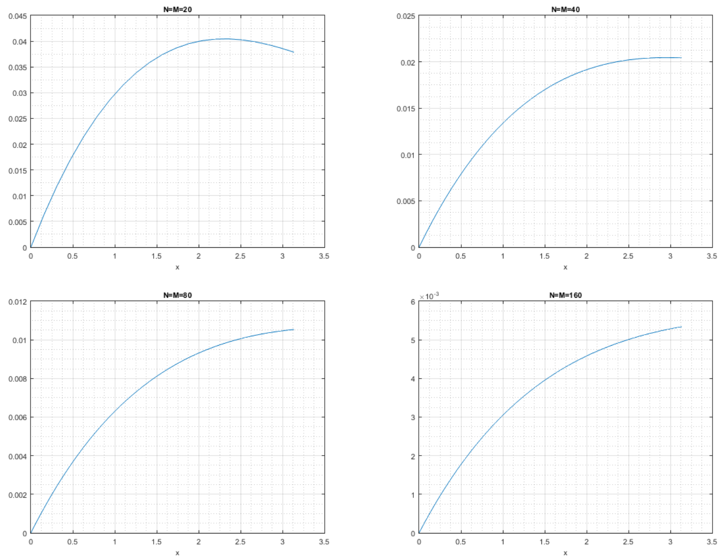

Figure 2: Graphs for errors of u in x.

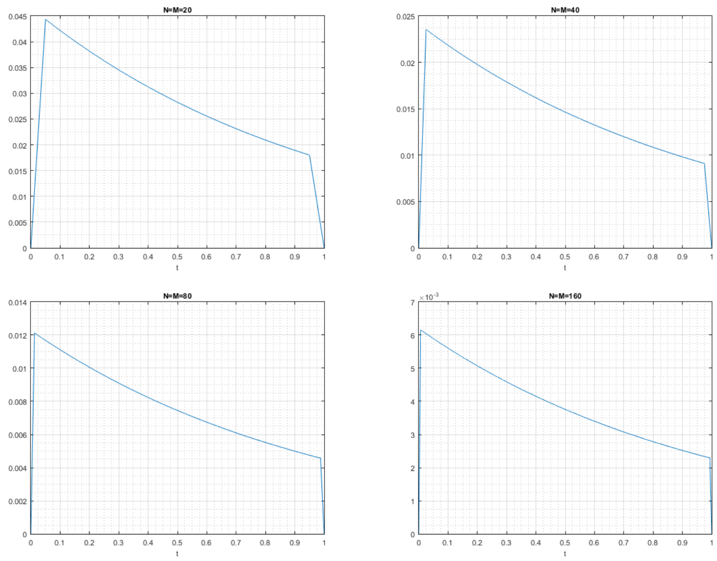

Figure 3: Graphs for errors of p in t.

5 Conclusion

In the present paper, the time-dependent identification problem for hyperbolic equation with periodic nonlocal conditions is studied. Applying a new version of a operator approach, we establish the well-posedness of the differential problem. A new absolute stable difference scheme for the approximate solution of this problem is constructed. The well-posedness of this difference scheme is established. Numerical test is given.

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